Interpolation Theory

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December 8, 2019

1 Reproducing Kernel Hilbert Spaces

Let X be a set and $\mathcal{H}$ a (complex) Hilbert space of functions defined on X, with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ such that the linear functionals $\ell_x : f \to f(x)$ are bounded. Such spaces are called reproducing kernel Hilbert space (or RKHS for short). The reason for this terminology is that by Riesz's representation Theorem there exists $k_x \in \mathcal{H}$ which represents ℓ_x, i.e.

$$f(x) = \langle f, k_x \rangle_{\mathcal{H}}, \ \ \forall f \in \mathcal{H}.$$

We can store all the information that k_x contain in a single function of two variables which we shall call *reproducing kernel* defined as follows

$$k(x, y) := k_y(x), \ \ x, y \in X.$$

We shall further assume that the Hilbert function spaces that we work with have the property that the kernel does not vanish on the diagonal.

Exercise 1.1. *Prove that if $k(x_0, x_0) = 0$ then $f(x_0) = 0, \ \forall f \in \mathcal{H}$.*

Lemma 1.1. *Suppose that k is a reproducing kernel the following hold.*

(i) $k(x, y) = \overline{k(y, x)}, \ \ x, y \in X$.

(ii) *k is positive semidefinite.*

These properties of a reproducing kernel characterize reproducing kernels completely.

Exercise 1.2. *Prove that if $k : X \times X \to \mathbb{C}$ with proeprties (i) and (ii) then there exists a reproducing kernel Hibert space of functions on X which has as reproducing kernel k.*

Example 1.1. *Let $H^2(\mathbb{D})$ be the Hardy space of analytic functions in the unit disc, that is functions f which have a power series expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \ \ z \in \mathbb{D},$$

with square summable coefficients. The Hardy space becomes a Hilbert space endowed with the inner product

$$\langle f, g \rangle := \sum_{n=0}^{\infty} a_n \overline{b_n},$$

where a_n, b_n are the Taylor coefficients of f and g respectively. In fact the Hardy space is a RKHS with reproducing kernel

$$\mathcal{S}(\zeta, \eta) = \frac{1}{1 - \overline{\eta}\zeta} \;\; \zeta, \eta \in \mathbb{D}.$$

The letter "S" for the reproducing kernel of the Hardy space comes from Szegö which is the name by which this kernel usually goes by.

2 Multipliers Space

For a given RKHS one can define the corresponding multiplier algebra, usually denoted by $\mathcal{M}(\mathcal{H})$ as the set

$$\{\varphi : X \to \mathbb{C}, \varphi \cdot f \in \mathcal{H}, \;\; \forall f \in \mathcal{H}\}.$$

For a given $\varphi \in \mathcal{M}(\mathcal{H})$ an application of the closed graph theorem gives that the operator

$$M_\varphi f := \varphi f,$$

is a bounded linear operator on $\mathcal{H}$. As a result there is a fundamental characterization of the multipliers space in purely operator theoretic terms.

Theorem 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$ (a bounded linear operator on $\mathcal{H}$). Then T is a multiplication operator if and only if every kernel vector k_x is an eigenvector of T^*.*

Proof. Suppose that $T = M_\varphi$ for some $\varphi \in \mathcal{M}(\mathcal{H})$. For any $f \in \mathcal{H}$ we have

$$\begin{aligned} \langle M_\varphi^* k_x, f \rangle_{\mathcal{H}} &= \langle k_x, M_\varphi f \rangle_{\mathcal{H}} \\ &= \langle k_x, \varphi f \rangle_{\mathcal{H}} \\ &= \overline{\varphi(x) f(x)} \\ &= \overline{\varphi(x)} \langle k_x, f \rangle_{\mathcal{H}} \\ &= \langle \overline{\varphi(x)} k_x, f \rangle_{\mathcal{H}}. \end{aligned}$$

Hence,

$$M_\varphi^* k_x = \overline{\varphi(x)} k_x. \tag{1}$$

The converse statement follows by the same calculation and the fact that $\vee\{k_x : x \in X\} = \mathcal{H}$. $\square$

Corollary 2.1. *For any $\varphi \in \mathcal{M}(\mathcal{H})$,*

$$\sup_{x\in X} |\varphi(x)| \leq \|M_\varphi\|.$$

Next we shall introduce tensor multipliers. This is done not only for the shake of generalizing the notion of a the multiplier space, but it turns out to be the right way to formulate a fundamental property of many RKHS.

Let μ be a at most countable cardinal, i.e. $\mu = 1, 2, \ldots, \aleph_0$. we denote by ℓ^2_μ either the Hilbert space $\mathbb{C}^\mu$ with the standard Hermitian inner product when μ is finite, or $\ell^2(\mathbb{N})$ when $\mu = \aleph_0$. Then the tensor product $\mathcal{H} \otimes \ell^2_\mu$ can be thought of as the space of column vectors

$$F = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \end{bmatrix}, \quad f_i \in \mathcal{H}, \;\; 1 \leq i \leq \mu \tag{2}$$

And norm given by

$$\|F\|^2_{\mathcal{H}\otimes\ell^2_\mu} := \sum_{i=1}^{\mu} \|f_i\|^2_{\mathcal{H}}$$

We define now tensor multipliers as follows, for any cardinals μ, ν as before the space of multipliers $\mathcal{M}(\mathcal{H} \otimes \ell^2_\mu, \mathcal{H} \otimes \ell^2_\nu)$ as functions $\Phi : X \to \mathcal{B}(\ell^2_\mu, \ell^2_\nu)$ such that

$$X \ni x \mapsto \Phi(x)F(x)$$

is in $\mathcal{H} \otimes \ell^2_\nu$ for all $F \in \mathcal{H} \otimes \ell^2_\mu$.

A similar statement as equation (1) holds for tensor multipliers,

$$M^*_\Phi(k_x \otimes v) = k_x \otimes \Phi^*(x)v, \;\; \forall x \in X, v \in \ell^2_\mu. \tag{3}$$

3 The Complete Nevanlinna Pick Property

The prototype for of all interpolation problems should probably be considered the Pick's interpolation problem. Suppose one is given $z_1, z_2, \ldots z_N$ points in $\mathbb{D}$ and $w_1, w_2, \ldots w_N$ complex number. What is a necessary and sufficient condition so that there exists $\varphi \in H^\infty(\mathbb{D})$, a bounded analytic function of supremum norm at most 1 such that

$$\varphi(z_i) = w_i, \;\; i = 1, \ldots N?$$

Instead of resolving the problem by an ad hoc method we shall try to formulate a much more general problem. The fundamental connection is the following observation left as an exercise.

Exercise 3.1. *It holds $\mathcal{M}(H^2(\mathbb{D})) = H^\infty(\mathbb{D})$ with equality of norms.*

Therefore Pick's problem can be seen as a problem of interpolation by multipliers of a RKHS. In this light one can formulate a general version of Pick's problem.

Suppose $\mathcal{H}$ is a RKHS on X and we are given a finite sequence of points $x_1, x_2, \dots x_N \in X$ and a bounded linear operators $W_1, W_2, \dots W_N \in \mathcal{B}(\ell^2_\mu, \ell^2_\nu)$ what is a necessary and sufficient condition such that there exists $\Phi \in \mathcal{M}(\mathcal{H} \otimes \ell^2_\mu, \mathcal{H} \otimes \ell^2_\nu)$ of operator norm at most 1 which interpolates the data, i.e.,

$$\Phi(x_i) = W_i, \;\; i = 1, \dots N?$$

In this generality we can formulate a necessary condition

Theorem 3.1. *Let $\mathcal{H}$ be a RKHS on X, let $x_1, x_2, \dots x_N \in X$ and let $W_1, W_2, \dots W_N \in \mathcal{B}(\ell^2_\mu, \ell^2_\nu)$. A necessary condition to be able to solve the corresponding Pick's interpolation problem is that the $\mathcal{B}(\ell^2_\nu)$-operator valued matrix*

$$\left[(I_\nu - W_i W_j^*) k(x_i, x_j)\right]_{i,j=1}^N \tag{4}$$

is positive semi-definite.

Proof. Suppose that such a Φ exists. As usual this is equivalent to

$$I_{\mathcal{H}\otimes\ell^2_\nu} - M_\Phi M_\Phi^* \geq 0$$

on $\mathcal{H} \otimes \ell^2\nu$. In particular if $v_1, v_2, \dots v_N \in \ell^2_\nu$,

$$\begin{aligned}
0 &\leq \Big\langle [I_{\mathcal{H}\otimes\ell^2_\nu} - M_\Phi M_\Phi^*]\big(\sum_{i=1}^N k_{x_i} \otimes v_i\big), \sum_{j=1}^N k_{x_j} \otimes v_j)\Big\rangle_{\mathcal{H}\otimes\ell^2_\nu} \\
&= \sum_{i,j=1}^N \Big[k(x_j, x_i)\langle v_i, v_j\rangle_{\ell^2_\nu} - \big\langle M_\Phi(k_{x_i} \otimes \Phi^*(x_i)v_i), k_{x_j} \otimes v_j\big\rangle_{\mathcal{H}\otimes\ell^2_\nu}\Big] \\
&= \sum_{i,j=1}^N \Big[k(x_j, x_i)\langle v_i, v_j\rangle_{\ell^2_\nu} - \big\langle k_{x_i} \otimes \Phi^*(x_i)v_i, k_{x_j} \otimes \Phi^*(x_j)v_j\big\rangle_{\mathcal{H}\otimes\ell^2_\nu}\Big] \\
&= \sum_{i,j=1}^N k(x_j, x_i)\langle (I_{\ell^2_\nu} - \Phi(x_j)\Phi^*(x_i))v_i, v_j)\rangle_{\ell^2_\nu}.
\end{aligned}$$

□

The previous condition is not always sufficient. But the cases when it is are so important that they deserve a definition

Definition 3.1. *We say that a RKHS $\mathcal{H}$ with reproducing kernel k has the $\mu \times \nu$ Nevanlinna-Pick property if condition (4) is also sufficient to solve the interpolating problem. If a kernel has the $\mu \times \nu$ Nevanlinna-Pick property we say that it is a Complete Nevanlinna-Pick Kernel.*

Let us now present mostly without proofs three different situations which are representative of the possible behaviours that one should expect.

Example 3.1 (The Paley Wiener Space)**.** *We say that an entire function f is of exponential type A if there exists a positive constant C such that $|f(z)| \leq Ce^{A|z|}, z \in \mathbb{C}$. It can be shown that for such functions, if $f|_{\mathbb{R}} \in L^2(\mathbb{R})$, the Fourier transform is supported on the interval $[-\pi, \pi]$, hence we can define the norm*

$$\|f\|^2_{PW^2_A} := \int_{-A}^{A} |\hat{f}(x)|^2 dx < +\infty.$$

It can be shown that that the space PW^2_A is a RKHS. We usually work take $A = \pi$. Then the reproducing kernel is given by

$$\sigma_\pi(z, w) := \frac{\sin \pi(z - \overline{w})}{\pi(z - \overline{w})}, \quad z, w \in \mathbb{C}.$$

By Corollary 2.1 if $\varphi \in \mathcal{M}(PW^2_\pi)$, it must be a bounded entire function therefore it should be constant. Therefore the Paley-Wiener spaces has only trivial multipliers.

This in particular implies that the The Paley Wiener space does not have the Pick Property.

Example 3.2 (The Bergman Space)**.** *The Bergman space $A^2(\mathbb{D})$ is the space of analytic functions in the unit disc which are square integrable with respect to the Lebesgue area measure $dA = \frac{dxdy}{\pi}$. Now let $\varphi \in H^\infty(\mathbb{D})$ and $f \in A^2(\mathbb{D})$,*

$$\int_{\mathbb{D}} |\varphi(z)f(z)|^2 dA(z) \leq \|\varphi\|^2_{H^\infty} \|f\|^2_{A^2}.$$

Hence,

$$\mathcal{M}(A^2(\mathbb{D})) = H^\infty(\mathbb{D}).$$

Although the multiplier algebra of the Bergman space contains a lot of non trivial elements, it turns out that they are not enough for the space to have the Nevanlinna-Pick property.

Suppose we want to solve the scalar interpolation problem for two points, so take for convenience $z_1 = w_1 = 0$, then the Pick matrix for the bergman kernel is positive semidefinite if and only if

$$|w_2| \leq |z_2|\sqrt{2 - |z_2|^2}$$

But we know that analytic functions in the unit ball of H^∞ reduce hyperbolic distance hence if such an interpolating function φ where to exist one should have

$$|w_2| = |\varphi(z_2)| \leq |z_2|$$

, which cleary it is not the case for all admissible choices of w_2.

It is a highly non trivial theorem that the Hardy space has the Complete Nevanlinna Pick Property. For a proof see for example [ref.]

4 Basics in Basis Theory in Hilbert Spaces

It will be useful for later to establish some terminology and present some basic results from the theory of bases in Hilbert spaces.

What are we going to talk about in this chapter makes sense in an arbitrary Hilbert space $\mathcal{H}$, even if a kernel structure is not specified.

Suppose we have a sequence of vectors $\{x_i\} \in \mathcal{H}$. For most of what is coming we assume that at least our sequence is *topologically free*, i.e.

$$x_i \notin \vee\{x_j : j \neq i\}.$$

Such systems always have what is called a dual system, that is a sequence $\{y_i\}$, with the property

$$\langle x_i, y_j \rangle_{\mathcal{H}} = \delta_{ij}, \forall i, j \in \mathbb{N}$$

In fact there exists a minimal dual system in the sense that the norms $\|y_i\|$ are the smallest possible.

Exercise 4.1. *Prove that if $\{y_i\}$ is the minimal dual system of $\{x_i\}$,*

$$\vee\{x_i\} = \vee\{y_i\}.$$

We can formally define an *analysis* operator associated to the sequence which maps an element $h \in \mathcal{H}$ to the sequence of its Fourier coefficients

$$\mathcal{F} : \mathcal{H} \to \ell^2, \;\; h \mapsto \{\langle h, x_i \rangle_{\mathcal{H}}\}_i.$$

The operator is densely defined on $\mathcal{H}$ because $\mathcal{H} = \vee\{y_i\} \oplus (\vee\{x_i\})^{\perp}$.

The (formal) adjoint of this operator is called *synthesis* operator and is given by

$$\mathcal{F}^* : \ell^2 \to \mathcal{H}, \;\; \{\alpha_i\} \mapsto \sum_i \alpha_i x_i.$$

This is in general only densely defined. Note also that both operators are closed.

Definition 4.1. *A sequence such that the associated synthesis operator is bounded it is called Bessel sequence. If $\mathcal{F}^*$ is also bounded below it is called Riesz sequence*

The matrix of the operator $G := \mathcal{F}\mathcal{F}^*$ with respect to the standard orthonormal basis of ℓ^2 is called the Grammian of the sequence. More explicitly,

$$G_{ij} = \langle x_i, x_j \rangle_{\mathcal{H}}.$$

We therefore have the following proposition.

Proposition 4.1. *For a topologically free system the following are equivalent.*

- *$\{x_i\}$ is a Bessel system.*
- *G is a bounded matrix in ℓ^2.*

- *The range of $\mathcal{F}$ is contained in ℓ^2.*

The proof of the following proposition is only slightly less trivial.

Proposition 4.2. *For a topologically free system the following are equivalent.*

- *The minimal dual system is a Bessel sequence.*
- *G is bounded below in ℓ^2.*
- *The range of $\mathcal{F}$ contains ℓ^2.*

Proof. Let $\{x_i\}$ a topologically free system, and $\{y_i\}$ its minimal dual. First we prove the equivalence of the first two elements in the list. Suppose $\{y_i\}$ is Bessel,

$$\begin{aligned}\|\sum_{i=1}^{N}\alpha_i x_i\|^2 &\geq \sup\Big\{\Big|\langle\sum\alpha_i x_i,\sum\beta_j y_j\rangle_{\mathcal{H}}\Big| : \|\sum\beta_j y_j\|\leq 1\Big\}\\ &\geq \sup\Big\{\Big|\sum\alpha_i\overline{\beta_i}\Big| : \Big(\sum|b_i|^2\Big)^{1/2}\leq\|G_\partial\|^{-1}\Big\}\\ &= \frac{1}{\|G_\partial\|}\sum_{i=1}^{N}|\alpha_i|^2.\end{aligned}$$

Note that this direction holds true even if $\{y_i\}$ is just **a** dual system of $\{x_i\}$. To see the other direction, the argument is the same with the role of x_i and y_i reversed noticing that we can reverse the inequalities because of minimality and the bounded below hypothesis.

Now we prove the equivalence of (1) and (3). Let $\{\alpha_i\}\in\ell^2$, then

$$\mathcal{F}\Big(\sum\alpha_i y_i\Big) = \{\alpha_i\}.$$

In the other direction, the open mapping principle allow us to construct a bounded right inverse of $\mathcal{F}$ which maps e_i t y_i.

$$\mathcal{F}R = \mathrm{Id}_{\ell^2},\;\; R:\ell^2\to\vee\{x_i\}\subseteq_{cl}\mathcal{H}.$$

In fact $R = \mathcal{F}_\partial^*$. It follows that the Grammian of $\{y_i\}$ is bounded and hence they are a Bessel system.

□

One of the most extraordinary theorems in the theory of bases in Hilbert spaces is Feichtinger's Theorem, a consequence of the solution of the Kadison-Singer problem, solved by Marcus-Spielman-Srivastava. We present it here without a proof.

Theorem 4.1 (Feichtinger's Theorem)**.** *Any Bessel sequence $\{x_i\}$ is a finite union of Riesz sequence.*

5 Interpolating Sequences

Let now $\mathcal{H}$ a RKHS,

Definition 5.1. *Let $\{x_i\}$ a sequence of points in X and $g_i := \frac{k_{x_i}}{\|k_{x_i}\|}$.*

- *We say that $\{x_i\}$ is Universally Interpolating (UI), if $\{g_i\}$ is a Riesz system in $\mathcal{H}$,*
- *Simply Interpolating (SI) if the minimal dual system of $\{g_i\}$ is Bessel,*
- *Carleson sequence (C) if the system $\{g_i\}$ is a Bessel system.*
- *Weakly Separated (WS) if it separated with respect to the Gleason metric $d_{\mathcal{H}}(x_i, x_j) := \sqrt{1 - |\langle g_i, g_j \rangle_{\mathcal{H}}|^2}$.*

Exercise 5.1. *Prove that the Gleason metric is actually a metric.*

Our next goal is to prove the following theorem

Theorem 5.1. *[Marshall, Sundberg, Aleman, McCarthy, Hartz] If $\mathcal{H}$ is a RKHS with the CNP property then a sequence $\{x_i\} \subseteq X$ is Universally Interpolating if and only if it is Weakly Separated and Carleson.*

Proof. First we prove the direct implication. For the converse we will need some more preparation. That a (UI) sequence is (C) is evident by definition, to see the (WS) part just let $\lambda \in \mathbb{C}, |\lambda| = 1$, by the Riesz basis property property for $i \neq j$

$$\varepsilon \leq \|g_i - \lambda g_j\|^2 = 2(1 - \overline{\lambda}\langle g_i, g_j \rangle_{\mathcal{H}}).$$

Taking infimum over all unimodular λ,

$$\frac{\varepsilon}{2} \leq 1 - |\langle g_i, g_j \rangle_{\mathcal{H}}|.$$

□

The following theorem is of fundamental importance and justifies the time we spent on tensor multipliers.

Theorem 5.2. *[Agler, McCarthy, Theorem 9.46] Let k a CNP kernel and $\{x_i\}$ a sequence of points, let g_i the corresponding normalized kernel vectors and let e_i be the standard orthonormal basis of ℓ^2.*

(a) $\{x_i\}$ is Simply Interpolating if and only if there exists a multiplier $\Psi \in \mathcal{M}(\mathcal{H} \otimes \ell^2, \mathcal{H})$ such that

$$\Psi(x_i) = e_i = \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots \end{pmatrix}.$$

Proof. If the sequence is Simple Interpolating by the discussion in the previous paragraph the associated Grammian G is bounded below, or equivalently there exists $\varepsilon > 0$ such that

$$G - \varepsilon I \geq 0.$$

Or to state it in a Pick matrix form,

$$[(1 - \varepsilon e_i \cdot e_j^*)k(x_i, x_j)] \geq 0.$$

Hence by the row Pick property there exists a multiplier $\tilde{\Phi} \in \mathcal{M}(\mathcal{H} \otimes \ell^2, \mathcal{H})$ of norm at most one, such that $\tilde{\Psi}(x_i) = \sqrt{\varepsilon} e_i$. Then $\Psi := \frac{\tilde{\Psi}}{\sqrt{\varepsilon}}$ is the desired multiplier.

The converse follows by the same argument, because the existence of such a multiplier implies the positivity of the Pick matrix. $\square$

Next theorem is an intermediate step in the proof, although of independent interest.

Theorem 5.3. *Let k a kernel with the two point scalar pick property and a $\{x_i\}$ Carleson and Weakly Separated sequence. Then there exist a sequence of multipliers $\theta_i \in \mathcal{M}_1(\mathcal{H})$ such that*

$$\theta_i(x_j) = \varepsilon \delta_{ij},$$

for some $\varepsilon > 0$.

Proof. Fix an $i \in \mathbb{N}$ and let $\phi_{ij} \in \mathcal{M}_1(\mathcal{H})$ such that,

$$\phi_{ij}(x_i) = 0, \phi_{ij}(x_j) = d_{\mathcal{H}}(x_i, x_j).$$

Such a matrix exists by positivity of the correspondent Pick matrix and the two point scalar pick property. The consider the multiplier $\theta_i \in \mathcal{M}_1(\mathcal{H})$

$$\theta_i := \prod_{j \neq i} \phi_{ij}.$$

(Check that the infinite product converges to a multiplier). Then θ_i vanishes on all points except x_j where it takes the value

$$\theta_i(x_i) = \prod_{j \neq i} d_{\mathcal{H}}(x_i, x_j).$$

Each factor is bounded away from zero by the Weak Separation condition, and also

$$\sum_{j \neq i}(1 - d_{\mathcal{H}}(x_i, x_j)) \leq 2 \sum_{j \neq i}(1 - d_{\mathcal{H}}(x_i, x_j)^2) = \sum_{j \neq i} |\langle g_i, g_j \rangle_{\mathcal{H}}|^2 \leq \|G\|_{\ell^2}^2.$$

Hence $\inf_{i \in \mathbb{N}} |\theta_i(z_i)| > 0$. $\square$

Proof of converse in Theorem 5.1. Let $\{x_i\}$ a sequence which is Weakly Separated and Carleson. By definition the system $\{g_i\}$ of normalized reproducing kernels forms a Bessel sequence, therefore by Feichtinger's Theorem it can be written as a finite union of Riesz systems or equivalently our sequence is a finite union of Universally Interpolating sequences. Therefore the claim will be proved if we show that the union of two (UI) sequences is (UI) if it is (WS).

We shall use the following notation. If $\{a_i\}, \{b_i\}$ are two infinite sequence we write $\{a_i\} \wedge \{b_i\}$ for the sequence

$$a_1, b_1, a_2, b_2, a_3, \ldots$$

Let $\{x_i^{(k)}\}, k = 1, 2$ be (UI) and $\{x_i\} := \{x_i^{(1)}\} \wedge \{x_i^{(2)}\}$ be (WS).

The union is also a Carleson sequence therefore there exist θ_i as in Theorem 5.3. Finally there exist multipliers $\Psi^{(1)}, \Psi^{(2)} \in \mathcal{M}(\mathcal{H} \otimes \ell^2, \mathcal{H})$ as in Theorem 5.2. Define the Ψ by

$$\Psi(x) := \left(\Psi^{(1)}(x) \wedge \Psi^{(2)}(x)\right) \begin{pmatrix} \theta_1(x) & 0 & 0 & \ldots \\ 0 & \theta_2(x) & 0 & \\ 0 & 0 & \theta_3(x) & \ldots \\ \vdots & & \vdots & \ddots \end{pmatrix}. \tag{5}$$

□

In fact $\Psi \in \mathcal{M}(\mathcal{H} \otimes \ell^2, \mathcal{H})$ and $\Psi(x_i) = e_i$. Hence, again by Theorem 5.2 the sequence is Universally Interpolating.